University of California, Berkeley
Physics 110A Fall 2001 Section 1 (*Strovink*)

Solution Set 6 (compiled by Daniel Larson)

1. **Griffiths 5.24** If $\mathbf{B}$ is uniform, then it is not a function of position, so any derivative of it vanish. In particular, $\boldsymbol{\nabla}\times\mathbf{B}=0$ and $\boldsymbol{\nabla}\cdot\mathbf{B}=0$. One can also check using cartesian coordinates that $\boldsymbol{\nabla}\times\mathbf{r}=0$ and $\boldsymbol{\nabla}\cdot\mathbf{r}=3$. Using these results we find $\boldsymbol{\nabla}\cdot\mathbf{A}=-\frac{1}{2}\boldsymbol{\nabla}\cdot(\mathbf{r}\times\mathbf{B})=-\frac{1}{2}[\mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{r})-\mathbf{r}\cdot(\boldsymbol{\nabla}\times\mathbf{B})]=0$. Also, using the fact that $(\mathbf{r}\cdot\boldsymbol{\nabla})\mathbf{B}=0$ since $\mathbf{B}$ is uniform, $\boldsymbol{\nabla}\times\mathbf{A}=-\frac{1}{2}\boldsymbol{\nabla}\times(\mathbf{r}\times\mathbf{B})=-\frac{1}{2}[(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{r}-(\mathbf{r}\cdot\boldsymbol{\nabla})\mathbf{B}+\mathbf{r}(\boldsymbol{\nabla}\cdot\mathbf{B})-\mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{r})]=-\frac{1}{2}[(\mathbf{B}\cdot)\mathbf{r}-3\mathbf{B}]$. Now, $(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{r}=\left(B_x\frac{\partial}{\partial x}+B_y\frac{\partial}{\partial y}+B_z\frac{\partial}{\partial z}\right)(x\,\hat{\mathbf{x}}+y\,\hat{\mathbf{y}}+z\,\hat{\mathbf{z}})=B_x\,\hat{\mathbf{x}}+B_y\,\hat{\mathbf{y}}+B_z\,\hat{\mathbf{z}}=\mathbf{B}$. Thus $\boldsymbol{\nabla}\times\mathbf{A}=-\frac{1}{2}(\mathbf{B}-3\mathbf{B})=\mathbf{B}$. We can add any constant to bfa without changing the divergence and curl, so the result is unique up to the addition of a constant vector field.

2. **Griffiths 5.25**

   (a) Let's assume that $\mathbf{A}$ points in the same direction as the current, namely the $\hat{\mathbf{z}}$ direction. Furthermore, the vector potential should be independent of ϕ and z because the infinite wire is symmetric with respect to translations and rotations about the z-axis. So we make the guess that $\mathbf{A}=A(s)\,\hat{\mathbf{z}}$. Using the formulas for taking divergence and curl in cylindrical coordinates, we find $\boldsymbol{\nabla}\cdot\mathbf{A}=\frac{\partial}{\partial z}A(s)=0$ and $\boldsymbol{\nabla}\times\mathbf{A}=-\frac{\partial}{\partial s}A(s)\,\hat{\boldsymbol{\phi}}$. Since $\mathbf{B}=\frac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}}$, we must have $\frac{\partial A}{\partial s}=-\frac{\mu_0 I}{2\pi s}\Rightarrow A(s)=-\frac{\mu_0 I}{2\pi}\ln s$. For the units to make sense, we need an arbitrary length in the logarithm, so finally $\mathbf{A}=-\frac{\mu_0 I}{2\pi}\ln(s/a)\,\hat{\mathbf{z}}$. (Note that putting "$a$" in the log is the same as adding a constant, so it doesn't change the divergence or curl of $\mathbf{A}$.)

   (b) First we need to find the magnetic field inside the wire, for $s<R$. Ampere's law gives $\oint\mathbf{B}\cdot d\mathbf{l}=2\pi sB(s)=\mu_0 I_{\text{enc}}=\mu_0 I\frac{\pi s^2}{\pi R^2}\Rightarrow\mathbf{B}=\frac{\mu_0 I s}{2\pi R^2}\,\hat{\boldsymbol{\phi}}$. We assume that $\mathbf{A}$ is of the same form as in part a, so $\frac{\partial A}{\partial s}=-\frac{\mu_0 I s}{2\pi R^2}\Rightarrow\mathbf{A}=-\frac{\mu_0 I}{4\pi R^2}(s^2-b^2)\,\hat{\mathbf{z}}$ where b is the constant of integration. For $s>R$ the B-field and thus $\mathbf{A}$ look the same as in part (a), except that we need $\mathbf{A}$ to be continuous at $s=R$. We can accomplish this by taking $a=b=R$. So finally, $\mathbf{A}=\begin{cases}-\frac{\mu_0 I}{4\pi R^2}(s^2-R^2)\,\hat{\mathbf{z}}, & \text{for } s\le R;\\ -\frac{\mu_0 I}{2\pi}\ln(s/R)\,\hat{\mathbf{z}}, & \text{for } s\ge R.\end{cases}$

3. **Griffiths 5.39**

   (a) Using the right-hand-rule, positive charges will be deflected down.

   (b) Charge accumulates on the bottom and top plates until the electric force balances the magnetic force. For a single charge, this means $qE=qvB\Rightarrow E=vB$. The field between two large, charged plates is essentially uniform, hence $V=Et$. So $V=vBt$. The bottom is at a higher potential, because that is where the positive charge is.

   (c) A current flowing to the right can be considered as positive charges flowing right or negative charges flowing left. If negative charges flow left, the will also feel a magnetic force downward, and thus *negative* charges will build up on the bottom plate. The potential difference between the top and bottom will be the same, but this time the top plate will be at higher potential.

4. **Griffiths 5.41** In cylindrical coordinates $\mathbf{B}$ is in the $\hat{\mathbf{z}}$ direction (either into or out of the page) and depends only on the radial distance s. The particle traveling in the shaded region is assumed to be in the $x-y$ plane at a location specified by the coordinate $\mathbf{r}$, with tangent vector $d\mathbf{l}=dr\,\hat{\mathbf{r}}+rd\phi\,\hat{\boldsymbol{\phi}}$. If the particle starts from the origin, it cannot have any angular momentum relative to the origin. If it emerges from the shaded region on a radial trajectory, its angular momentum is $\mathbf{r}\times\mathbf{p}=0$. So if we can show that the particle acquires no angular momentum throughout its motion, we will have proven that it must emerge on a radial trajectory. We also know that $\int\mathbf{B}\cdot d\mathbf{a}=\int\mathbf{B}2\pi r\,dr=0$. Recall that the torque about the origin is $\mathbf{N}=\frac{d\mathbf{L}}{dt}=\mathbf{r}\times\mathbf{F}$.

$$\mathbf{L}=\int\frac{d\mathbf{L}}{dt}\,dt=\int(\mathbf{r}\times\mathbf{F})\,dt=\int\mathbf{r}\times q(\mathbf{v}\times\mathbf{B})\,dt=q\int\mathbf{r}\times(d\mathbf{l}\times\mathbf{B})=q\left[\int(\mathbf{r}\cdot\mathbf{B})\,d\mathbf{l}-\int\mathbf{B}(\mathbf{r}\cdot d\mathbf{l})\right],$$

where we have used $\mathbf{v}dt = d\mathbf{l}$ and the BAC-CAB rule for a triple cross product. Now, since the particle is in the xy-plane and $\mathbf{B}$ is normal to the page, $\mathbf{r}\cdot\mathbf{B} = 0$. Also, $\mathbf{r}\cdot d\mathbf{l} = r\,\hat{\mathbf{r}}\cdot(dr\,\hat{\mathbf{r}} + rd\phi\,\hat{\boldsymbol{\phi}}) = r\,dr$. So $\mathbf{L} = -\frac{q}{2\pi}\int \mathbf{B}2\pi r\,dr = 0$ because $B_x = B_y = 0$ and $\int B_z 2\pi r\,dr = 0$ by assumption. Thus the particle emerges with zero total angular momentum, which means it must be traveling along a radial line.

5. **Griffiths 5.56**

   (a) The angular momentum of a ring is $\mathbf{L} = I\omega\,\hat{\mathbf{z}}$ with $I = MR^2$, and its dipole moment will be $\mathbf{m} = IA\,\hat{\mathbf{z}} = \frac{Q}{2\pi/\omega}\pi R^2\,\hat{\mathbf{z}} = \frac{1}{2}Q\omega R^2\,\hat{\mathbf{z}}$. Thus $\mathbf{m} = \frac{Q}{2M}\mathbf{L}$. So the gyromagnetic ration is $g = \frac{Q}{2M}$.

   (b) Because g is independent of the radius, the same applies to all infinitesimal rings of charge. We could calculate the total angular momentum of a spinning sphere by adding up the contributions from each ring, just as we could get the total magnetic moment by adding up the contributions from each ring. Since each ring will contribute to the magnetic moment and angular momentum in the same proportion, the ratio of total dipole moment to angular momentum will be the same as in part (a), $g = \frac{Q}{2M}$.

   (c) If the electron has angular momentum $\frac{1}{2}\hbar$ then the dipole moment m will be

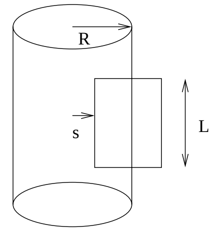
$$m = \frac{e}{2m_e}\frac{1}{2}\hbar = \frac{e\hbar}{4m_e} = \frac{(1.60\times10^{-19}\text{ C})(1.05\times10^{-34}\text{ Js})}{4(9.11\times10^{-31}\text{ kg})} = 4.61\times10^{-24}\text{ A m}^2.$$

6. **Griffiths 6.10** Because the magnetization is uniform, $\boldsymbol{\nabla}\times\mathbf{M} = 0$, so there is no volume bound current, but only a surface bound current $K_b = M$, wrapping around the rod like the current in a solenoid. For $a \ll L$, a is much smaller than the radius of the toroid, so in equation (5.58), we can treat s as the radius of the toroid. Then $\frac{NI}{2\pi s}$ is the amount of current flowing around the toroid, per unit length, which is exactly what we mean by surface current. Thus the B-field inside a complete, magnetized toroid is $\mathbf{B} = \mu_0\frac{NI}{2\pi s}\,\hat{\boldsymbol{\phi}} = \mu_0 K_b\,\hat{\boldsymbol{\phi}} = \mu_0\mathbf{M}$.

   But part of the toroid is cut out, which we can treat as a bunch of square loops carrying the opposite current; hence they will produce a magnetic field in a direction opposite to the one produced by the rest of the toroid. In problem (5.8) we found the B-field at the center of a square loop: $B = \mu_0 I\sqrt{2}/\pi R$. In this case $R = a/2$ (the perpendicular distance from the center of the loop to its side). We assume that $w \ll a$, so we can think of the gap as a single square loop with all the current running around it. Thus $I = K_b w = Mw$. So the missing piece of the toroid contributes $-2\sqrt{2}\mu_0 Mw/\pi a$. So at the center of the gap, $\mathbf{B} = \mu_0\mathbf{M}\Big(1 - \frac{2\sqrt{2}w}{\pi a}\Big)$.

7. **Griffiths 6.12**

   (a) There is a surface bound current $\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} = kR\,\hat{\boldsymbol{\phi}}$ and a volume current $\mathbf{J}_b = \boldsymbol{\nabla}\times\mathbf{M} = -k\,\hat{\boldsymbol{\phi}}$. Since all the current is circumferential, we can think of the situation as the superposition of lots of coaxial solenoids of different radii. So immediately we conclude $\mathbf{B} = 0$ outside the cylinder. Now we can draw a square amperian loop that has one side parallel to the z-axis inside the cylinder, and the opposite side parallel to the z-axis outside. We know the B-field should be pointing in the z-direction, so we'll get no contribution to the line integral from the other two sides. Since $B = 0$ outside, the only section of the loop that contributes is the piece inside the cylinder parallel to the z-axis. $\oint \mathbf{B}\cdot d\mathbf{l} = BL = \mu_0 I_{\text{enc}} = \mu_0\big[\int J_b da + K_b L\big] = \mu_0[-kL(R-s) + kRL] = \mu_0 kLs$. ($L(R-s)$ is the area of the amperian loop inside the cylinder.) So $\mathbf{B} = \mu_0 ks\,\hat{\mathbf{z}}$ inside.



Problem 7. Griffiths 6.12

(b) Since $\mathbf{M}$ is the only object in this problem that picks out a direction in space, we know $\mathbf{H}$ must also point in the z-direction. However, using the same amperian loop as in part (a), $\oint \mathbf{H} \cdot d\mathbf{l} = HL = \mu_0 I_{f_{\text{enc}}} = 0$ because there are no free currents. Thus $\mathbf{H} = 0$, so $\mathbf{B} = \mu_0 \mathbf{M}$. Outside, $\mathbf{M} = 0$ so $\mathbf{B} = 0$; inside $\mathbf{M} = ks\,\hat{\mathbf{z}}$, so $\mathbf{B} = \mu_0 ks\,\hat{\mathbf{z}}$.

8. **Griffiths 6.13** We assume that the cavities as small enough so that the fields are essentially uniform inside of them. We treat the cavities by considering the superposition of a piece of material without cavities and small, cavity-shaped objects with opposite magnetization.

(a) The B-field of a uniformly magnetized sphere is $\frac{2}{3}\mu_0 \mathbf{M}$, so the contribution to the B-field from the cavity is the same as the contribution from a uniformly magnetized sphere with magnetization $-\mathbf{M}$, namely $\mathbf{B}_{\text{cav}} = -\frac{2}{3}\mu_0 \mathbf{M}$. Thus with the sphere removed $\mathbf{B} = \mathbf{B}_0 - \frac{2}{3}\mu_0 \mathbf{M}$. Inside the real cavity, $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}$ because there is no magnetization, so $= \frac{1}{\mu_0}(\mathbf{B_0} - \frac{2}{3}\mu_0 \mathbf{M}) = \mathbf{H_0} + \mathbf{M} - \frac{2}{3}\mathbf{M} \Rightarrow \mathbf{H} = \mathbf{H_0} + \frac{1}{3}\mathbf{M}$.

(b) For a long, thin, cylindrical cavity with uniform magnetization $-\mathbf{M}$ there is only surface current $K_b = -M$, which looks like a solenoid. So the B-field at the center is $\mu_0 K_b = -\mu_0 M$. Adding this to the contribution from the cavity-less material, we find $\mathbf{B} = \mathbf{B}_0 - \mu_0 \mathbf{M}$. Then $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} = \frac{1}{\mu_0}(\mathbf{B}_0 - \mu_0 \mathbf{M}) = \frac{1}{\mu_0}\mathbf{B}_0 - \mathbf{M} \Rightarrow \mathbf{H} = \mathbf{H}_0$.

(c) For the wafer shaped cavity, the bound currents run around the outside edge, so if the wafer has a large radius and is very thin, those currents will be very small and far away from the center and will contribute virtually no magnetic field. Thus $\mathbf{B} = \mathbf{B}_0$. Then $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}_0 = \mathbf{H}_0 + \mathbf{M} \Rightarrow \mathbf{H} = \mathbf{H}_0 + \mathbf{M}$.